On Markov–Duffin–Schaeffer inequalities with a majorant. II

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Abstract

We are continuing out studies of the so-called Markov inequalities with a majorant. Inequalities of this type provide a bound for the k-th derivative of an algebraic polynomial when the latter is bounded by a certain curved majorant μ. A conjecture is that the upper bound is attained by the so-called snake-polynomial which oscillates most between $\pm\mu$, but it turned out to be a rather difficult question.

In the previous paper, we proved that this is true in the case of symmetric majorant provided the snake-polynomial has a positive Chebyshev expansion. In this paper, we show that that the conjecture is valid under the condition of positive expansion only, hence for non-symmetric majorants as well.

1 Introduction

This paper continues our studies in [7] and it is dealing with the problem of estimating the max-norm $\|p^{(k)}\|$ of the k-th derivative of an algebraic polynomial p of degree n under restriction

$$|p(x)| \le \mu(x), \qquad x \in [-1,1],$$

where μ is a non-negative majorant. We want to find for which majorants μ the supremum of $\|p^{(k)}\|$ is attained by the so-called snake-polynomial ω_μ which oscillates most between $\pm\mu$, namely by the polynomial of degree n that satisfies the following conditions

$$a) \quad |\omega_\mu(x)| \le \mu(x), \qquad b) \quad \omega_\mu(\tau_i^*) = (-1)^i \mu(\tau_i^*), \quad i = 0, \dots, n\,.$$

(This is an analogue of the Chebyshev polynomial T_n for $\mu \equiv 1$.) Actually, we are interested in those μ that provide the same supremum for $\|p^{(k)}\|$ under the weaker assumption

$$|p(x)| \le \mu(x), \qquad x \in \delta^* = (\tau_i^*)_{i=0}^n,$$

where δ^* is the set of oscillation points of ω_μ. These two problems are generalizations of the classical results for $\mu \equiv 1$ of Markov [4] and of Duffin-Schaeffer [2], respectively.

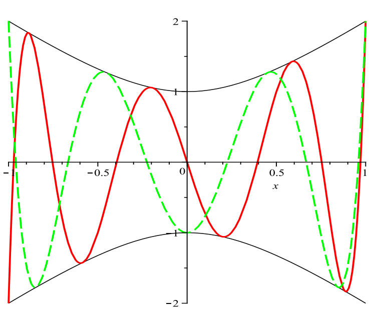

Fig. 1. Markov inequality with a majorant μ:

$|p| \le \mu, \quad \|p^{(k)}\| \to \sup$

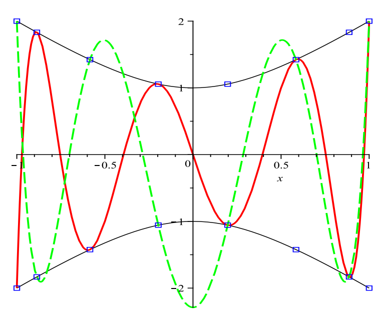

Fig. 2. Duffin-Schaeffer inequality with a majorant μ:

$|p|_{\delta^*} \le |\mu|_{\delta^*}, \quad \|p^{(k)}\| \to \sup$

Problem 1.1 (Markov inequality with a majorant) For $n, k \in \mathbb{N}$, and a majorant $\mu \ge 0$, find

$$M_{k,\mu} := \sup_{|p(x)| \le \mu(x)} \|p^{(k)}\| \tag{1.1}$$

Problem 1.2 (Duffin–Schaeffer inequality with a majorant) For $n, k \in \mathbb{N}$, and a majorant $\mu \ge 0$, find

$$D^*_{k,\mu} := \sup_{|p|_{\delta^*} \le |\mu|_{\delta^*}} \|p^{(k)}\| \tag{1.2}$$

In these notation, results of Markov [4] and Duffin–Schaeffer [2] read:

$$\mu \equiv 1 \quad \Rightarrow \quad M_{k,\mu} = D^*_{k,\mu} = \|T_n^{(k)}\|\,,$$

so, the question of interest is for which other majorants μ the snake-polynomial ω_μ is extremal to both problems (1.1)-(1.2), i.e., when we have the equalities

$$M_{k,\mu} \stackrel{?}{=} D^*_{k,\mu} \stackrel{?}{=} \|\omega_\mu^{(k)}\|\,. \tag{1.3}$$

Note that, for any majorant μ, we have $\|\omega_\mu^{(k)}\| \le M_{k,\mu} \le D^*_{k,\mu}$, so the question marks in (1.3) will be removed once we show that

$$D^*_{k,\mu} \le \|\omega_\mu^{(k)}\|\,. \tag{1.4}$$

Ideally, we would also like to know the exact numerical value of $\|\omega_\mu^{(k)}\|$ and that requires some kind of explicit expression for the snake-polynomial ω_μ. The latter is available for the class of majorants of the form

$$\mu(x) = \sqrt{R_s(x)}, \tag{1.5}$$

where R_s is a non-negative polynomial of degree s, so it is this class that we paid most of our attention to.

In the previous paper [7], we proved that inequality (1.4) is valid if $\widehat{\omega}_\mu := \omega_\mu^{(k-1)}$ belongs to the class Ω which is defined by the following three conditions:

$$\widehat{\omega}_\mu \in \Omega: \quad \begin{array}{lll} 0) & \widehat{\omega}_\mu(x) = \prod_{i=1}^{\widehat{n}}(x - t_i), \qquad t_i \in [-1,1]; & \\ 1a) & \|\widehat{\omega}_\mu\|_{C[0,1]} = \widehat{\omega}_\mu(1), & 1b) \quad \|\widehat{\omega}_\mu\|_{C[-1,0]} = |\widehat{\omega}_\mu(-1)|; \\ 2) & \widehat{\omega}_\mu = c_0 + \sum_{i=1}^{\widehat{n}} a_i T_i\,, \quad a_i \ge 0. & \end{array}$$

Theorem 1.3 ([7]) *Let $\omega_\mu^{(k-1)} \in \Omega$. Then*

$$M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)\,.$$

Let us make some comments about the polynomial class Ω.

For ω_μ, assumption (0) is redundant, as the snake-polynomial $\omega_\mu \in \mathcal{P}_n$ has $n+1$ points of oscillations between $\pm\mu$, hence, all of its n zeros lie in the interval $[-1,1]$, thus the same is true for any of its derivative. We wrote it down as we will use this property repeatedly.

In the case of symmetric majorant μ, condition (1) becomes redundant too, as in this case the snake-polynomial ω_μ is either even or odd, hence all T_i in its Chebyshev expansion (2) are of the same parity, and that, coupled with non-negativity of a_i, implies (1a) and (1b).

Corollary 1.4 *Let $\mu(x) = \mu(-x)$, and let ω_μ be the corresponding snake-polynomial of degree n. If*

$$\omega_\mu^{(k_0)} = c_0 + \sum_{i=1}^{\widehat{n}} a_i T_i\,, \quad a_i \ge 0,$$

then

$$M_{k,\mu} = D_{k,\mu} = \omega_\mu^{(k)}(1)\,, \qquad k \ge k_0 + 1\,.$$

This corollary allowed us to establish Duffin-Schaeffer (and, thus, Markov) inequalities for various symmetric majorants μ of the form (1.5).

However, for non-symmetric ω_μ satisfying (2), equality (1b) is often not valid for small k, and that did not allow us to bring our Duffin-Schaeffer-type results to a satisfactory level. For example, (1b) is not fulfilled in the case

$$\mu(x) = x + 1, \qquad k = 1,$$

although intuitively it is clear that the Duffin-Schaeffer inequality with such μ should be true.

Here, we show that, as we conjectured in [7]), inequality (1.4) is valid under condition (2) only, hence, the statement of Corollary 1.4 is true for non-symmetric majorants μ as well.

Theorem 1.5 *Given a majorant $\mu \ge 0$, let ω_μ be the corresponding snake-polynomial of degree n. If*

$$\omega_\mu^{(k_0)} = c_0 + \sum_{i=1}^{\widehat{n}} a_i T_i \,, \quad a_i \ge 0,$$

then

$$M_{k,\mu} = D_{k,\mu} = \omega_\mu^{(k)}(1)\,, \qquad k \ge k_0 + 1\,.$$

A short proof of this theorem is given in § 3. It is based on a new idea which allow to "linearize" the problem and reduce it to the following property of the Chebyshev polynomial T_n.

Proposition 1.6 *For a fixed $t \in [-1, 1]$, let*

$$\tau_n(x, t) := \frac{1 - xt}{x - t}(T_n(x) - T_n(t))\,.$$

Then

$$\max_{x,t\in[-1,1]} |\tau_n'(x, t)| = T_n'(1).$$

A simple and explicit form of the polynomials $\tau_n'(x, t)$ involved allows to draw their graphs in a straightforward way and thus to check this proposition numerically for rather large degrees n. The graphs below show that $\tau_n'(x, t)$, as a function of two variables, has $n - 3$ local extrema approximately half the value of the global one, namely

$$\max_{|x|\le\cos\frac{\pi}{n}} \max_{|t|\le1} |\tau_n'(x, t)| \approx \frac{1}{2} T_n'(1)\,,$$

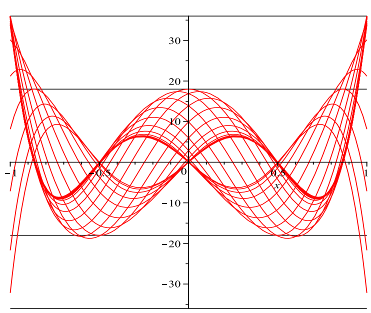

Fig. 3. Graphs of $\tau_n'(\cdot, t)$ for $n = 6$

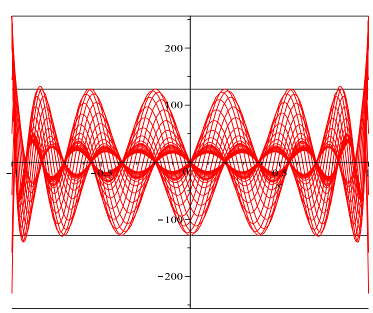

Fig. 4. Graphs of $\tau_n'(\cdot, t)$ for $n = 16$

However, the rigorous proof of Proposition 1.6 turned out to be relatively long, and it would be interesting to find a shorter one.

2 Markov-Duffin-Schaeffer inequalities for various majorants

1) Before our studies, Markov- or Duffin-Schaeffer-type inequalities were obtained for the following majorants μ and derivatives k:

Markov-type inequalities: $M_{k,\mu} = \omega_\mu^{(k)}(1)$

1°	$\sqrt{ax^2+bx+1},\ b \ge 0$	$k=1$	[16]	2°	$(1+x)^{\ell/2}(1-x^2)^{m/2}$	$k \ge m + \frac{l}{2}$	[8]
3°	$\sqrt{1+(a^2-1)x^2}$	all k	[16]	4°	$\sqrt{\prod_{i=1}^m (1+c_i^2 x^2)}$	$k=1$	[17]

Duffin-Schaeffer-type inequalities: $M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)$

2_1^*	$\sqrt{1-x^2}$	$k \ge 2$	[10]	2_2^*	$1-x^2$	$k \ge 3$	[11]

Our next theorem combines results from our previous paper [7] with some new results based on Theorem 1.5. In particular, it shows that, in cases 1° and 4°, Markov-type inequalities $M_{k,\mu} = \omega_\mu^{(k)}(1)$ are valid also for $k \ge 2$, and in case 2° they are valid for $k \ge m+1$ independently of ℓ. Moreover, in all our cases we have stronger Duffin-Schaeffer-type inequalities.

Theorem 2.1 *Let μ be one of the majorant given below. Then, with the corresponding k, the $(k-1)$-st derivative of its snake-polynomial ω_μ satisfies*

$$\omega_\mu^{(k-1)} = \sum_i a_i T_i, \qquad a_i \ge 0\,, \tag{2.1}$$

hence, by Theorem 1.5,

$$M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)\,. \tag{2.2}$$

Duffin-Schaeffer-type inequalities: $M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)$

1^*	$\sqrt{ax^2+bx+1},\ b \ge 0$	$k \ge 2$	new	2^*	$(1+x)^{\ell/2}(1-x^2)^{m/2}$	$k \ge m+1$	new
3^*	$\sqrt{1+(a^2-1)x^2}$	$k \ge 2$	[7]	4^*	$\sqrt{\prod_{i=1}^m (1+c_i^2 x^2)}$	$k \ge 1$	[7]
5^*	any $\sqrt{R_m(x^2)}$	$k \ge m+1$	[7]	6^*	any $\mu(x) = \mu(-x)$	$k \ge \lfloor \frac{n}{2} \rfloor + 1$	[7]
7^*	$\sqrt{(1+c^2x^2)(1+(a^2-1)x^2)}$	$k \ge 2$	[7]	8^*	$\sqrt{1-a^2x^2+a^2x^4}$	$k \ge 1$	new

Proof. The proof of (2.1) for particular majorants consists of sometimes tedious checking.

a) The cases 3^*-7^*, with symmetric majorants μ, are taken from [7] where we already proved both (2.1) and (2.2). Here, we added one more symmetric case 8^* as an example of the majorant which is not monotonely increasing on $[0,1]$, but which is still providing Duffin-Schaeffer inequality for all $k \ge 1$. (One can check that its snake-polynomial has the form $\omega_\mu(x) = bT_{n+2} + (1-b)T_{n-2}$.

b) In the non-symmetric case 1^*, we also proved (2.1) for $k \ge 2$ already in [7], however in [7] we were able to get (2.2) only for $k \ge 3$.

c) The second non-symmetric case 2^* is new, but proving (2.1) in this case is relatively easy. □

2) Our next theorem allows to produce Duffin-Schaeffer inequalities for various types of majorants based on the cases that have been already established.

Theorem 2.2 *Let a majorant μ have the form*

$$\mu(x) = \mu_1(x)\mu_2(x) := \sqrt{Q_r(x)}\sqrt{R_s(x)},$$

where the snake-polynomials for μ_1 and μ_2, respectively, satisfy

$$\omega_{\mu_1}^{(m_1)} = \sum a_i T_i, \quad a_i \ge 0, \qquad \omega_{\mu_2}^{(m_2)} = \sum b_i T_i, \quad b_i \ge 0\,.$$

Then the snake-polynomial for μ satisfies

$$\omega_{\mu}^{(m_1+m_2)} = \sum c_i T_i, \quad c_i \ge 0\,.$$

In the following example, 9^* is a combination of the cases 2^* (with $m = 0$) and 4^*, and 10^* is a combination of 1^* with itself.

Further Markov-Duffin-Schaeffer inequalities: $M_{k,\mu} = D^*_{k,\mu} = \omega_\mu^{(k)}(1)$

9^*	$(1+x)^{l/2}\sqrt{\prod_{i=1}^m (1+c_i^2x^2)}$	$k \ge 1$		10^*	$\sqrt{\prod_{i=1}^m (a_ix^2+b_ix+1)},\ b_i \ge 0$	$k \ge m+1$

In fact, cases 2^*, 4^* and 7^* can be obtained in the same way from the majorants of degree 1 and 2.

3) There are two particular cases of a majorant μ and a derivative k for which Markov-type inequalities have been proved, but which cannot be extended to Duffin-Schaeffer-type within our method, as in those case $\omega_\mu^{(k-1)}$ does not have a positive Chebyshev expansion.

Markov- but not Duffin-Schaeffer-type inequalities: $M_{k,\mu} = \omega_\mu^{(k)}(1), \quad D^*_{k,\mu} = ?$

1°	$\sqrt{ax^2+bx+1},\ b \ge 0$	$k = 1$		2°	$(1-x^2)^{m/2}$	$k = m$

In this respect, a natural question is whether this situation is due to imperfectness of our method, or maybe it is because the equality $M_{k,\mu} = D^*_{k,\mu}$ is no longer valid. An indication that the latter could undeed be the case was obtained by Rahman-Schmeisser [10] for the majorant $\mu_1(x) := \sqrt{1-x^2}$. Namely they showed that

$$\mu_1(x) = \sqrt{1-x^2}, \quad k = 1 \quad \Rightarrow \quad 2n = \omega'_{\mu_1}(1) = M_{1,\mu_1} < D^*_{1,\mu_1} = \mathcal{O}(n\ln n)\,.$$

Here, we show that, in case 2°, i.e., for $\mu_m := (1-x^2)^{m/2}$ with any m, similar inequalities between Markov and Duffin-Schaeffer constants hold for all $k \le m$.

Theorem 2.3 *We have*

$$\mu_m(x) = (1-x^2)^{m/2}, \quad k \le m \quad \Rightarrow \quad \mathcal{O}(n^k) = M_{k,\mu_m} < D^*_{k,\mu_m} = \mathcal{O}(n^k \ln n)\,.$$

As to the remaining case 1°, we believe that if $\mu(1) > 0$, i.e., except the degenerate case $\mu(x) = \sqrt{1-x^2}$, we will have Markov-Duffin-Schaeffer inequality at least for large n:

$$\mu(x) = \sqrt{ax^2+bx+1}, \quad b \ge 0, \quad k = 1 \quad \Rightarrow \quad M_{1,\mu} = D_{1,\mu} = \omega'_\mu(1), \qquad n \ge n_\mu\,,$$

where n_μ depends on $\mu(1)$.

3 Proof of Theorem 1.5

In [7], we used the following intermediate estimate as an upper bound for $D^*_{k,\mu}$.

Proposition 3.1 ([7]) *Given a majorant μ, let $\omega_\mu \in \mathcal{P}_n$ be its snake-polynomial, let $\widehat{\omega}_\mu(x) := \omega_\mu^{(k-1)}(x)$, and let*

$$\phi_{\widehat{\omega}}(x, t_i) := \frac{1 - xt_i}{x - t_i}\, \widehat{\omega}_\mu(x), \qquad \textit{where } t_i \textit{ are the roots of } \widehat{\omega}_\mu. \tag{3.1}$$

Then

$$D^*_{k,\mu} \le \max\left\{\omega_\mu^{(k)}(1),\ \max_{x,t_i\in[-1,1]} |\phi'_{\widehat{\omega}}(x,t_i)|\right\}. \tag{3.2}$$

We showed in [7] that if $\widehat{\omega}_\mu \in \Omega$, then $\phi'_{\widehat{\omega}}(x,t_i)| \le \widehat{\omega}'_\mu(1) = \omega_\mu^{(k)}(1)$.

Here, we will use very similar estimate.

Proposition 3.2 *Given a majorant μ, let $\omega_\mu \in \mathcal{P}_n$ be its snake-polynomial, let $\widehat{\omega}_\mu = \omega_\mu^{(k-1)}$, and let*

$$\tau_{\widehat{\omega}}(x,t) := \frac{1-xt}{x-t}(\widehat{\omega}_\mu(x) - \widehat{\omega}_\mu(t)), \qquad t \in [-1,1]. \tag{3.3}$$

Then

$$D^*_{k,\mu} \le \max\left\{\omega_\mu^{(k)}(1),\ \max_{x,t\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t)|\right\}. \tag{3.4}$$

Proof. Comparing two definitions (3.1) and (3.3), we see that, since $\widehat{\omega}(t_i) = 0$, we have

$$\tau_{\widehat{\omega}}(x,t_i) = \frac{1-xt_i}{x-t_i}(\widehat{\omega}_\mu(x) - \widehat{\omega}_\mu(t_i)) = \frac{1-xt_i}{x-t_i}\,\widehat{\omega}_\mu(x) = \widehat{\phi}_{\widehat{\omega}}(x,t_i)\,.$$

Therefore,

$$\max_{x,t_i\in[-1,1]} |\phi'_{\widehat{\omega}}(x,t_i)| = \max_{x,t_i\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t_i)| \le \max_{x,t\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t)|\,,$$

and (3.4) follows from (3.2).

Proof of Theorem 1.5. By Proposition 3.2, we are done if we prove that

$$\max_{x,t\in[-1,1]} |\tau'_{\widehat{\omega}}(x,t)| \le \widehat{\omega}'_\mu(1) \quad \left(= \omega_\mu^{(k)}(1)\right).$$

By assumption,

$$\widehat{\omega}_\mu = c_0 + \sum_{i=1}^{\widehat{n}} a_i T_i, \qquad a_i \ge 0, \tag{3.5}$$

therefore

$$\begin{aligned} \widehat{\tau}_{\widehat{\omega}}(x,t) &:= \frac{1-xt}{x-t}(\widehat{\omega}_\mu(x) - \widehat{\omega}_\mu(t)) = \frac{1-xt}{x-t}\sum_{i=1}^{\widehat{n}} a_i(T_i(x) - T_i(t)) \\ &= \sum_{i=1}^{\widehat{n}} a_i \frac{1-xt}{x-t}(T_i(x) - T_i(t)) = \sum_{i=1}^{\widehat{n}} a_i \tau_i(x,t)\,, \end{aligned}$$

and respectively

$$|\tau'_{\widehat{\omega}}(x,t)| \le \sum_{i=1}^{\widehat{n}} |a_i||\tau'_i(x,t)| \overset{(a)}{=} \sum_{i=1}^{\widehat{n}} a_i|\tau'_i(x,t)| \overset{(b)}{\le} \sum_{i=1}^{\widehat{n}} a_i T'_i(1) \overset{(c)}{=} \widehat{\omega}'_\mu(1).$$

Here, the equality (a) is due to assumption $a_i \ge 0$ in (3.5), equality (c) also follows from (3.5), and inequality (b) is the matter of the Proposition 1.6.

4 Preliminaries

For a polynomial

$$\omega(x) = c\prod_{i=1}^{n}(x-t_i)\,, \quad -1 \le t_n \le \cdots \le t_1 \le 1, \quad c>0,$$

with all its zeros in the interval $[-1,1]$ (and counted in the reverse order), set

$$\phi(x,t_i) := \frac{1-xt_i}{x-t_i}\,\omega(x)\,, \qquad i=1,\ldots,n. \tag{4.1}$$

For each i, we would like to estimate the norm $\|\phi'(\cdot,t_i)\|_{C[-1,1]}$, i.e., the maximum value of $|\phi(x,t_i)|$, and the latter is attained either at the end-points $x=\pm 1$, or at the points x where $\phi''(x,t_i)=0$.

Let us introduce two functions

$$\psi_1(x,t) \;:=\; \frac{1}{2}(1-xt)\,\omega''(x) - t\,\omega'(x)\,. \tag{4.2}$$

$$\psi_2(x,t) \;:=\; \frac{1}{2}(1-x^2)\,\omega''(x) + \frac{x-t}{1-xt}\,\omega'(x) - \frac{x(1-t^2)}{(x-t)(1-xt)}\,\omega(x)\,. \tag{4.3}$$

In [7] we obtained the following results.

Claim 4.1 *We have*

$$|\phi'(\pm 1,t_i)| \le |\omega'(\pm 1)|\,.$$

Claim 4.2 *For each i, both $\psi_{1,2}(\cdot,t_i)$ interpolate $\phi'(\cdot,t_i)$ at the points of its local extrema,*

$$\phi''(x,t_i)=0 \quad\Rightarrow\quad \phi'(x,t_i)=\psi_{1,2}(x,t_i)\,, \tag{4.4}$$

Claim 4.3 *With some specific functions $f_\nu(\omega,\cdot)$, we have*

$$1)\quad |\psi_1(x,t_i)| \le \max_{\nu=1,2,3}|f_\nu(x)|, \qquad 0\le x\le 1, \qquad -1\le \tfrac{x-t_i}{1-xt_i}\le \tfrac{1}{2}\,;$$

$$2)\quad |\psi_2(x,t_i)| \le \max_{\nu=1,2}|f_\nu(x)|, \qquad t_1\le x\le 1; \qquad \tfrac{1}{2}\le \tfrac{x-t_i}{1-xt_i}\le 1;$$

and, under addittional assumption that $|\omega(x)|\le\omega(1)$,

$$3)\quad |\psi_2(x,t_i)| \le \max_{\nu=1,2,4}|f_\nu(x)|, \qquad 0\le x\le t_1, \qquad \tfrac{1}{2}\le \tfrac{x-t_i}{1-xt_i}\le 1.$$

Claim 4.4 *Let*

$$\omega = c_0 + \sum_{i=1}^{n} a_i T_i, \qquad a_i\ge 0\,,$$

Then

$$\max_{1\le\nu\le 4}|f_\nu(\omega,x)| \le \omega'(1)\,.$$

From Claims 4.1-4.4, we obtain the following theorem.

Theorem 4.5 *Let ω satisfy the following three conditions*

$$0)\qquad \omega(x) = c\prod_{i=1}^{n}(x-t_i) \qquad t_i\in[-1,1],$$

$$1a)\qquad \|\omega\|_{C[0,1]} = \omega(1), \qquad\qquad 1b)\quad \|\omega\|_{C[-1,0]} = |\omega(-1)|;$$

$$2)\qquad \omega = c_0 + \sum_{i=1}^{n} a_i T_i, \quad a_i\ge 0\,.$$

Then

$$\max_{x,t_i\in[-1,1]} |\phi'(x,t_i)| \le \omega'(1)$$

We will need the following corollary.

Proposition 4.6 *Let*

$$\omega(x) = c_0 + T_n(x) = \prod_{i=1}^n (x - t_i), \quad |c_0| \le 1,$$

and let a pair of points (x, t_i) satisfy any of the following conditions:

$$\begin{array}{lll} 1) & 0 \le x \le 1, & -1 \le \frac{x-t_i}{1-xt_i} \le \frac{1}{2}\,; \\ 2) & t_1 \le x \le 1; & \frac{1}{2} \le \frac{x-t_i}{1-xt_i} \le 1; \\ 3) & 0 \le x \le t_1, & \frac{1}{2} \le \frac{x-t_i}{1-xt_i} \le 1 \quad \textit{and} \quad |\omega(x)| \le \omega(1). \end{array} \tag{4.5}$$

Then

$$\phi''(x,t_i) = 0 \quad \Rightarrow \quad |\phi'(x,t_i)| \le \omega'(1) \tag{4.6}$$

5 Preliminaries

Here, we will prove Proposition 1.6, namely that the polynomial

$$\tau(x,t) := \frac{1-xt}{x-t}\,(T_n(x) - T_n(t))\,,$$

considered as a polynomial in x, admits the estimate

$$|\tau'(x,t)| \le T_n'(1)\,, \qquad x,t \in [-1,1]\,, \qquad n \in \mathbb{N}\,. \tag{5.1}$$

We prove it as before by considering, for a fixed t, the points x of local extrema of $\tau'(x,t)$ and the end-points $x = \pm 1$, and showing that at those points $|\tau'(x,t)| \le T_n'(1)$.

Lemma 5.1 *If $x = \pm 1$, then $|\tau'(x,t)| \le T_n'(1)$.*

Proof. This inequality follows from the straightforward calculations:

$$\tau'(1,t) = T_n'(1) - \frac{1+t}{1-t}\,(T_n(1) - T_n(t))\,.$$

The last term is non-negative, hence $\tau'(1,t) \le T_n'(1)$. Also, since $1+t \le 2$ and $\frac{T_n(1)-T_n(t)}{1-t} \le T_n'(1)$, it does not exceed $2T_n'(1)$, hence $\tau'(1,t) \ge -T_n'(1)$.

It remains to consider the local maxima of $\tau'(\cdot,t)$, i.e., the points (x,t) where $\tau_n''(x,t) = 0$ Note that local maxima of the polynomial $\tau_n'(\cdot,t)$ exist only for $n \ge 3$, and that, because of symmetry $\tau(x,t) = \pm\tau(-x,-t)$, it is sufficient to prove the inequality (5.1) only on the half of the square $[-1,1] \times [-1,1]$. So, we are dealing with the case

$$\mathcal{D}: \quad x \in [0,1], \quad t \in [-1,1]\,; \qquad n \ge 3.$$

We split the domain $\mathcal{D}$ into two main subdomains:

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2, \qquad \begin{array}{llll} \mathcal{D}_1: & x \in [0,1], & t \in [-1,1], & -1 \le \frac{x-t}{1-xt} \le \frac{1}{2}\,; \\ \mathcal{D}_2: & x \in [0,1], & t \in [-1,1], & \frac{1}{2} \le \frac{x-t}{1-xt} \le 1\,; \end{array}$$

with a further subdivision of $\mathcal{D}_2$

$$\mathcal{D}_2 = \mathcal{D}_2^{(1)} \cup \mathcal{D}_2^{(2)} \cup \mathcal{D}_2^{(3)}, \qquad \begin{array}{llll} \mathcal{D}_2^{(1)}: & x \in [0,1], & t \in [\cos\frac{3\pi}{2n}, 1], & \frac{1}{2} \le \frac{x-t}{1-xt} \le 1; \\ \mathcal{D}_2^{(2)}: & x \in [0, \cos\frac{\pi}{n}], & t \in [-1, \cos\frac{3\pi}{2n}], & \frac{1}{2} \le \frac{x-t}{1-xt} \le 1; \\ \mathcal{D}_2^{(3)}: & x \in [\cos\frac{\pi}{n}, 1], & t \in [-1, \cos\frac{3\pi}{2n}], & \frac{1}{2} \le \frac{x-t}{1-xt} \le 1. \end{array}$$

We will prove

Proposition 5.2 *a) if $(x,t) \in \mathcal{D}_1 \cup \mathcal{D}_2^{(1)} \cup \mathcal{D}_2^{(2)}$ and $\tau''(x,t) = 0$, then $|\tau'(x,t)| \le T_n'(1)$;*
b) if $(x,t) \in \mathcal{D}_2^{(3)}$, then $\tau''(x,t) \ne 0$.

For (a), we use use results of §4, in particular Proposition 4.6.

6 Proof of Proposition 5.2.a

Proposition 6.1 *For a fixed $t \in [-1,1]$, let t_1 be the rightmost zero of the polynomial*

$$\omega_*(\cdot) = T_n(\cdot) - T_n(t)\,,$$

and let a pair of points (x,t) satisfy any of the following conditions:

$$\begin{array}{llll} 1') & 0 \le x \le 1, & -1 \le \frac{x-t}{1-xt} \le \frac{1}{2}\,; & \\ 2') & t_1 \le x \le 1; & \frac{1}{2} \le \frac{x-t}{1-xt} \le 1; & \\ 3') & 0 \le x \le t_1, & \frac{1}{2} \le \frac{x-t}{1-xt} \le 1 & \text{and} \quad T_n(t) \le 0\,. \end{array} \tag{6.1}$$

Then

$$\tau''(x,t) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le T_n'(1)\,. \tag{6.2}$$

Proof. For a fixed $t \in [-1,1]$, the polynomial $\omega_*(\cdot) = T_n(\cdot) - T_n(t)$ has n zeros inside $[-1,1]$ counting possible multiplicities, i.e. $\omega_*(x) = c\prod(x - t_i)$, and $x = t$ is one of them, i.e., $t = t_i$ for some i. Therefore, conditions $(1')$-$(3')$ for (x,t) in (6.1) are equaivalent to the conditions (1)-(3) for (x, t_i) in (4.5), in particular, the inequality $|\omega_*(x)| < \omega_*(1)$ in 4.5(3) follows from $T_n(t) \le 0$. Hence, the implication (4.6) for ϕ_* is valid. But, since $t = t_i$, we have

$$\tau(x,t) = \frac{1-xt}{x-t}\,(T_n(x) - T_n(t)) = \frac{1-xt_i}{x-t_i}\,\omega_*(x) = \phi_*(x,t_i),$$

so (6.2) is identical to (4.6).

Lemma 6.2 *Let $(x,t) \in \mathcal{D}_1 = \{x \in [0,1], t \in [-1,1], -1 \le \frac{x-t}{1-xt} \le \frac{1}{2}]\}$. Then*

$$\tau''(x,t_i) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le T_n'(1)\,.$$

Proof. Condition $(x,t) \in \mathcal{D}_1$ is identical to condition (1′) in Proposition 6.1, hence the conclusion.

Lemma 6.3 *Let $(x,t) \in \mathcal{D}_2^{(1)} = \{x \in [0,1], t \in [\cos\frac{3\pi}{2n}, 1], -1 \le \frac{x-t}{1-xt} \le \frac{1}{2}]\}$. Then*

$$\tau''(x,t_i) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le T_n'(1)\,.$$

Proof. We split $\mathcal{D}_2^{(2)}$ into two further sets:

$$2a)\quad t \in [\cos\tfrac{3\pi}{2n}, \cos\tfrac{\pi}{2n}]\,, \qquad 2b)\quad t \in [\cos\tfrac{\pi}{2n}, 1]\,.$$

2a) For $t \in [\cos\frac{3\pi}{2n}\cos\frac{\pi}{2n}]$ we have $T_n(t) \le 0$, so we apply again Proposition 6.1 where we use condition (3′), if $x < t_1$, and condition (2′) otherwise.

2b) For $t \in [\cos\frac{\pi}{2n}, 1]$, the Chebyshev polynomial $T_n(t)$ is increasing, hence t is the rightmost zero t_1 of the polynomial $\omega_*(x) = T_n(x) - T_n(t)$. Now, we use the inequality $\frac{1}{2} \le \frac{x-t}{1-xt} \le 1$ for $(x,t) \in \mathcal{D}_2^{(2)}$. Since $t = t_1$, we have

$$\frac{1}{2} \le \frac{x-t_1}{1-xt_1} \le 1 \quad \Rightarrow \quad t_1 \le x \le 1,$$

so we apply Proposition 6.1 with condition (2′).

Lemma 6.4 *Let* $(x,t) \in \mathcal{D}_2^{(2)} = \{x \in [0, \cos\frac{\pi}{n}], t \in [-1, \cos\frac{3\pi}{2n}], -1 \le \frac{x-t}{1-xt} \le \frac{1}{2}]\}$*. Then*

$$\tau''(x,t) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le T_n'(1)\,.$$

Proof. By Claim 4.2, since $\tau(x,t) = \phi_*(x,t_i)$, we have

$$\tau''(x,t) = 0 \quad \Rightarrow \quad |\tau'(x,t)| \le |\psi_2(x,t)|\,,$$

where

$$\psi_2(x,t) := \frac{1}{2}(1-x^2)\,\omega_*''(x) + \frac{x-t}{1-xt}\,\omega_*'(x) - \frac{x(1-t^2)}{(x-t)(1-xt)}\,\omega_*(x)\,, \tag{6.3}$$

so let us prove that

$$\max_{x,t\in\mathcal{D}_2^{(2)}} |\psi_2(x,t)| \le T_n'(1). \tag{6.4}$$

Making the substitution $\gamma = \frac{x-t}{1-xt}$ into (6.3), we obtain

$$\psi_2(x,t) := \psi_\gamma(x) = \frac{1}{2}(1-x^2)\,\omega_*''(x) + \gamma\,\omega_*'(x) - \frac{1-\gamma^2}{\gamma}\,\frac{x}{1-x^2}\,\omega_*(x) \;\; =: \; g_\gamma(x) - h_\gamma(x)\,, \tag{6.5}$$

where $g_\gamma(x)$ is the sum of the first two terms, and $h_\gamma(x)$ is the third one, so that

$$|\psi_2(x,t_i)| \le |g_\gamma(x)| + |h_\gamma(x)| \tag{6.6}$$

Let us evaluate both g_γ and h_γ.

1) Since $\omega_*(x) = T_n(x) - T_n(t)$, we have

$$2g_\gamma(x) = (1-x^2)T_n''(x) + 2\gamma T_n'(x) = (x+2\gamma)T_n'(x) - n^2T_n(x)\,,$$

so that, using Cauchy inequality and the well-known identity for Chebyshev polynomials, we obatin

$$\begin{aligned} 2|g_\gamma(x)| &= n\left|nT_n(x) - \frac{x+\gamma}{n\sqrt{1-x^2}}\sqrt{1-x^2}T_n'(x)\right| \\ &\le n\left(n^2T_n(x)^2 + (1-x^2)T_n'(x)^2\right)^{1/2}\left(1 + \frac{(x+2\gamma)^2}{n^2(1-x^2)}\right)^{1/2} \\ &\le n^2\left(1 + \frac{(x+2\gamma)^2}{n^2(1-x^2)}\right)^{1/2} \end{aligned} \tag{6.7}$$

2) For the function h_γ, since $\omega_*(x) = T_n(x) - T_n(t)$ does not exceed 2 in the absolute value, we have the trivial estimate

$$|h_\gamma(x)| \le \frac{1-\gamma^2}{\gamma}\,\frac{2x}{1-x^2} \;= n^2\frac{1-\gamma^2}{\gamma}\,\frac{2x}{n^2(1-x^2)}\,. \tag{6.8}$$

3) So, from (6.6), (6.7) and (6.8), we have

$$\max_{x,t\in(C_3)} |\psi_2(x,t_i)| \le T_n'(1)\max_{x,\gamma} F(x,\gamma)\,,$$

where

$$F(x,\gamma) := \frac{1}{2}\left(1 + \frac{(x+2\gamma)^2}{n^2(1-x^2)}\right)^{1/2} + \frac{1-\gamma^2}{\gamma}\,\frac{2x}{n^2(1-x^2)},$$

and the maximum is taken over $\gamma \in [\frac{1}{2}, 1]$ and $x \in [0, x_n]$, $x_n = \cos\frac{\pi}{n}$. Clearly, $F(x,\gamma) \le F(x_n,\gamma)$, so we are done with (6.4) once we prove that $F(x_n,\gamma) \le 1$. We have

$$\begin{aligned}
F(x_n,\gamma) &= \frac{1}{2}\left(1 + \frac{(\cos\frac{\pi}{n}+2\gamma)^2}{n^2\sin^2\frac{\pi}{n}}\right)^{1/2} + \frac{1-\gamma^2}{\gamma}\,\frac{2\cos\frac{\pi}{n}}{n^2\sin^2\frac{\pi}{n}} \\
&\le \frac{1}{2}\left(1 + \frac{(1+2\gamma)^2}{4^2\sin^2\frac{\pi}{4}}\right)^{1/2} + \frac{1-\gamma^2}{\gamma}\,\frac{2\cdot 1}{4^2\sin^2\frac{\pi}{4}} \;=:\; G(\gamma), \qquad n \ge 4,
\end{aligned}$$

where we used $\cos\frac{\pi}{n} < 1$ and the fact that the sequence $(n^2\sin^2\frac{\pi}{n})$ is increasing. Hence, $F(x_n,\gamma) \le 1$ for all $n \ge 3$ if

$$F(x_3,\gamma) \le 1, \qquad G(\gamma) \le 1$$

and the latter follows from the graphs

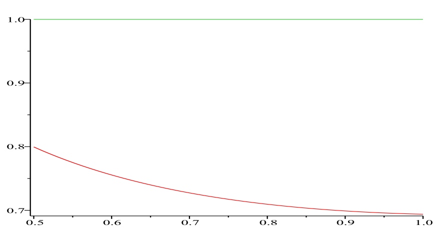

Figure 1: The graph of $F(x_3,\gamma)$

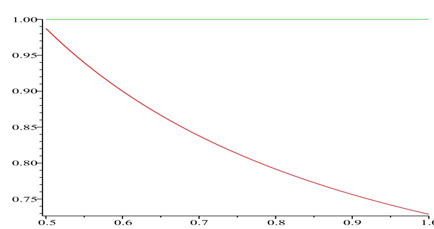

Figure 2: The graphs of $G(\gamma)$

7 Proof of Proposition 5.2.b

Lemma 7.1 *Let* $x \in \mathcal{D}_2^{(3)} = \{x \in [\cos\frac{\pi}{n}, 1], t \in [-1, \cos\frac{3\pi}{2n}], \frac{1}{2} \le \frac{x-t}{1-xt} \le 1\}$. *Then*

$$\tau''(x,t) > 0\,.$$

We prove this statement in several steps, restriction $\frac{1}{2} \le \frac{x-t}{1-xt} \le 1$ is irrelevant.

Lemma 7.2 *a) If* $t \in [-1, 0]$, *then* $\tau''(x,t) > 0$ *for* $x \ge \cos\frac{\pi}{n}$.

b) If $t \in (0, 1]$, *then* $\tau''(x,t)$ *has at most one zero on* $[\cos\frac{\pi}{n}, \infty)$,

and $\tau''(x,t) < 0$ *for large* x.

Proof. By definition,

$$\tau(x,t) = \frac{1-xt}{x-t}\,(T_n(x) - T_n(t))\ .$$

For a fixed $t \in [-1,1]$, the polynomial $\omega_*(\cdot) = T_n(\cdot) - T_n(t)$ has n zeros inside $[-1,1]$, say (t_i), one of them at $x = t$, so $t = t_i$ for some i. From definition, we see that the polynomial $\tau(\cdot,t)$ has the same zeros as ω_* except t_i which is replaced by $1/t_i$. So, if $(s_i)_{i=1}^n$ an $(t_i)_{i=1}^n$ are the zeros of $\tau(\cdot,t)$ and $\omega_*(\cdot,t)$ respectively, counted in the reverse order, then

$$1)\quad s_i \le t_i \le s_{i-1}, \quad \text{if} \quad t \le 0, \qquad 2)\quad s_{i+1} \le t_i \le s_i, \quad \text{if} \quad t > 0\,.$$

That means that zeros of $\tau(\cdot,t)$ and $\omega_*(\cdot)$ interlace, hence, by Markov's lemma, the same is true for the zeros of any of their derivatives. In particular, for the rightmost zeros of the second derivatives, we have

$$1)\quad s_1'' < t_1'', \quad \text{if} \quad t \le 0, \qquad 2)\quad s_2'' < t_1'' < s_1'', \quad \text{if} \quad t > 0\,.$$

Since $\omega_*'' = T_n''$, its rightmost zero t_1'' satisfies $t_1'' < \cos\frac{\pi}{n}$ as the latter is the rightmost zero of T_n', This proves case (a) and the forst part of the case (b) of the lemma. Second part of (b_2) follows from the observation that, for $t > 0$, polynomial $\tau(\cdot,t)$ has a negative leading coefficient.

Corollary 7.3 *For a fixed $t \in [0,1]$, if $\tau''(x,t) \ge 0$ at $x = 1$, then $\tau''(x,t) > 0$ for all $x \in [\cos\frac{\pi}{n}, 1)$.*

Lemma 7.4 *If $t \in [0, \cos\frac{3\pi}{2n}]$, then $\tau''(x,t) > 0$ for $x \in [x_n, 1]$*

Proof. We have

$$\tau''(x,t) = \frac{1-xt}{x-t}\,T_n''(x) - 2\,\frac{1-t^2}{(x-t)^2}\,T_n'(x) + 2\,\frac{1-t^2}{(x-t)^3}\,(T_n(x) - T_n(t))$$

By the previous corollary, it is sufficient to prove that

$$\tau''(1,t) = \frac{n^2(n^2-1)}{3} - 2\,\frac{1+t}{1-t}\,n^2 + 2\,\frac{1+t}{(1-t)^2}\,(1 - T_n(t)) \ge 0\,. \tag{7.1}$$

1) Since the last term is non-negative for $t \in [-1,1)$, this inequality is true if

$$\frac{n^2(n^2-1)}{3} - 2\,\frac{1+t}{1-t}\,n^2 \ge 0 \quad \Rightarrow \quad t \le \frac{n^2-7}{n^2+5}\,.$$

We have

$$\cos\frac{3\pi}{2n} < \frac{n^2-7}{n^2+5}\,, \quad 4 \le n \le 6\,, \quad \text{and} \quad \cos\frac{2\pi}{n} < \frac{n^2-7}{n^2+5} < \cos\frac{3\pi}{2n}\,, \quad n \ge 7.$$

So, we are done, once we prove that (7.1) is valid for $t \in [\cos\frac{2\pi}{n}, \cos\frac{3\pi}{2n}]$ and $n \ge 7$.

2) Consider the function

$$f(t) := (1-t)\tau''(1,t) = (1-t)\frac{n^2(n^2-1)}{3} - 2(1+t)n^2 + 2\,(1+t)\,\frac{1-T_n(t)}{1-t}\,.$$

This function is convex on $I = [\cos\frac{2\pi}{n}, +\infty]$. Indeed, the first two terms are linear in t and the last term consists of two factors, both convex, positive and increasing on I. The latter claim is obvious for $1+t$, and it is true for $P_n(t) := \frac{1-T_n(t)}{1-t}$, since this P_n is a polynomial with positive leading coefficient whose rightmost zero is the double zero at $t = \cos\frac{2\pi}{n}$.

So, f is convex and satisfies $f(0) = 0$, $f(\cos\frac{2\pi}{n}) > 0$, therefore if $f(t_*) > 0$ for some t_*, then $f(t) > 0$ for all $t \in [\cos\frac{2\pi}{n}, t_*]$.

Thus, it remains to show that $\tau''(1, \cos\frac{3\pi}{2n}) > 0$, i.e.,

$$\frac{n^2(n^2-1)}{3} - 2n^2 u + \frac{2}{1+\cos\frac{3\pi}{2n}} u^2 > 0, \quad u = \cot^2\frac{3\pi}{4n}\,. \tag{7.2}$$

This inequality will certainly be true if $u^2 - 2n^2 u + \frac{n^2(n^2-1)}{3} > 0$, and a sufficient condition for the latter is

$$\cot^2\frac{3\pi}{4n} = u < n^2\Big(1 - \sqrt{\frac{2}{3} + \frac{1}{3n^2}}\Big)$$

Since $\cot\alpha < \alpha^{-1}$ for $0 < \alpha < \frac{\pi}{2}$, this condition is fulfilled if $(\frac{4}{3\pi})^2 < 1 - \sqrt{\frac{2}{3} + \frac{1}{3n^2}}$ and that is true for $n \ge 8$. For $n = 7$, we verify (7.2)directly.

8 Proof of Theorem 2.2

Lemma 8.1 *Let a majorant μ have the form $\mu(x) = \sqrt{R_s(x)}$, where R_s is a non-negative polynomial of degree s. Then, for $N \ge \lfloor -\frac{s+1}{2} \rfloor$, its snake-polynomial ω_N of degree $N+s$ has the form*

$$\omega_\mu = \sum_{i=0}^{s} a_i T_{N+i}$$

Lemma 8.2 *Let a majorant μ have the form*

$$\mu(x) = \mu_1(x)\mu_2(x) = \sqrt{Q_r(x)}\sqrt{R_s(x)}$$

and let

$$\omega_{\mu_1} = \sum_{i=0}^{r} a_i T_{N+i}, \qquad \omega_{\mu_2} = \sum_{i=0}^{s} b_i T_{N+i}\,.$$

Then

$$\omega_\mu = \sum_{i=0}^{r}\sum_{j=0}^{s} a_i b_j T_{N+i+j}$$

Proof of Theorem 2.2.

9 Proof of Theorem 2.3

In this section, we prove that, for the majorant $\mu_m(x) = (1-x^2)^{m/2}$, its snake-polynomial ω_μ is not extremal for the Duffin-Schaeffer inequality for $k \le m$, i.e., for

$$D^*_{k,\mu_m} := \sup_{|p(x)|_{\delta^*} \le |\mu_m(x)|_{\delta^*}} \|p^{(k)}\|$$

where $\delta^* = (\tau_i^*)$ is the set of points of oscillation of ω_μ between $\pm\mu_m$, we have

$$D^*_{k,\mu_m} > \|\omega_\mu^{(k)}\|, \qquad k \le m.$$

Snake-polynomial for μ is given by the formula

$$\omega_{\mu_m}(x) = \begin{cases} (x^2-1)^s T_n(x), & m = 2s, \\ (x^2-1)^s \frac{1}{n} T_n'(x), & m = 2s-1\,, \end{cases}$$

so its oscillation points are the sets

$$\delta_n^1 := (\cos \tfrac{\pi i}{n})_{i=0}^n\,, \qquad \delta_n^2 := (\cos \tfrac{\pi(i-1/2)}{n})_{i=1}^n\,,$$

where $|T_n(x)| = 1$ and $|T_n'(x)| = \frac{n}{\sqrt{1-x^2}}$, respectively, with additional multiple points at $x = \pm 1$. Now, we introduce the pointwise Duffin-Schaeffer function:

$$d_{k,\mu}^*(x) := \sup_{|p|_{\delta^*} \le |\mu_m|_{\delta^*}} |p^{(k)}(x)| = \begin{cases} \sup\limits_{|q|_{\delta^1} \le |T_n|_{\delta^0}} |[(x^2-1)^s q(x)]^{(k)}|\,, & m = 2s, \\ \sup\limits_{|q|_{\delta^2} \le \frac{1}{n}|T_n'|_{\delta^1}} |[(x^2-1)^s q(x)]^{(k)}|\,, & m = 2s-1\,, \end{cases}$$

and note that

$$D_{k,\mu}^* = \|d_{k,\mu}^*(\cdot)\| \ge d_{k,\mu}^*(0)\,.$$

Proposition 9.1 *We have*

$$D_{k,\mu_m}^* \ge \mathcal{O}(n^k \ln n)\,.$$

Proof. We divide the proof in two cases, for even and odd m, respectively.

Case 1 ($m = 2s$). Let us show that, for a fixed $k \in \mathbb{N}$, and for all large $n \not\equiv k \,(\mathrm{mod}\, 2)$, there is a polynomial $q_1 \in \mathcal{P}_n$ such that

$$1) \quad |q_1(x)|_{\delta_n^1} \le 1, \qquad 2) \quad |[(x^2-1)^s q_1(x)]^{(k)}|_{x=0} = \mathcal{O}(n^k \ln n)\,.$$

1) Set

$$P(x) := (x^2-1)T_n'(x) = c\prod_{i=0}^n (x - t_i)\,, \qquad (t_i)_{i=0}^n = (\cos \tfrac{\pi i}{n})_{i=0}^n = \delta_n^1\,,$$

and, having in mind that $t_{n-i} = -t_i$, define the polynomial

$$q_1(x) := \frac{1}{n^2} P(x) \sum_{i=1}^{(n-1)/2} \Big(\frac{1}{x - t_i} - \frac{1}{x + t_i}\Big) =: \frac{1}{n^2} P(x) U(x)\,.$$

This polynomial vanishes at all t_i that do not appear under the sum, i.e., at $t_0 = 1$, $t_n = -1$ and, for even n, at $t_{n/2} = 0$. At all other t_i it has the absolute value $|q(t_i)| = \frac{1}{n^2}|P'(t_i)| = 1$, by virtue of $P'(x) = n^2 T_n(x) + xT_n'(x)$.

2) We see that U is even, and P is either even or odd, and for their nonvanishing derivatives at $x = 0$ we have

$$\begin{aligned} |P^{(r)}(0)| &= |T_n^{(r+1)}(0) - r(r+1)T_n^{(r-1)}(0)| = \mathcal{O}(n^{r+1})\,, \quad n \not\equiv r \,(\mathrm{mod}\, 2)\,, \\ |U^{(r)}(0)| &= 2r! \sum_{i=1}^{(n-1)/2} \frac{1}{(t_i)^{r+1}} = \sum_{j=1}^{(n-1)/2} \frac{1}{(\sin \frac{\pi j}{n})^{r+1}} = \begin{cases} \mathcal{O}(n \ln n), & r = 0, \\ \mathcal{O}(n^{r+1}), & r = 2r_1 \ge 2. \end{cases} \end{aligned}$$

Respectively, in Leibnitz formula for $q_1^{(k)}(x) = \frac{1}{n^2}[P(x)U(x)]^{(k)}$, the term $P^{(k)}(0)U(0) = \mathcal{O}(n^{k+2} \ln n)$ dominates, hence

$$q_1^{(k)}(0) = \mathcal{O}(n^k \ln n) \quad \Rightarrow \quad [(x^2-1)q(x)]_{x=0}^{(k)} = \mathcal{O}(n^k \ln n)\,.$$

Case 2 ($m = 2s-1$). Similarly, for a fixed k, and for all large $n \equiv k \,(\mathrm{mod}\, 2)$, the polynomial $q_2 \in \mathcal{P}_{n-1}$ defined as

$$q_2(x) := \frac{1}{n} T_n(x) \sum_{i=1}^{(n-1)/2} \Big(\frac{1}{x - t_i} - \frac{1}{x + t_i}\Big), \qquad (t_i)_{i=1}^n = (\cos \tfrac{\pi(i-1/2)}{n})_{i=1}^n = \delta_n^2\,,$$

satisfies

$$1) \quad |q_2(x)|_{\delta_n^1} \le \frac{1}{n}|T_n'(x)|_{\delta_n^1}\,, \qquad 2) \quad |(x^2-1)^s q_2^{(k)}(x)|_{x=0} = \mathcal{O}(n^k \ln n)\,.$$

Proposition 9.2 *Let $\mu_m(x) = (1-x^2)^{m/2}$. Then*

$$M_{k,\mu_m} = \mathcal{O}(n^k), \quad k \le m\,. \tag{9.1}$$

Proof. Pierre and Rahman [8] proved that

$$M_{k,\mu_m} := \sup_{|p(x)|\le|\mu_m(x)|} \|p^{(k)}\| = \max\left(\|\omega_N^{(k)}\|, (\|\omega_{N-1}^{(k)}\|\right), \qquad k \ge m\,,$$

where ω_N and ω_{N-1} are the snake-polynomial for μ_m of degree N and $N-1$, respectively. However, they did not investigate which norm is bigger and at what point $x \in [-1,1]$ it is attained. We proved in [7] that, for $f(x) := (x^2-1)^s T_n(x)$ and for $g(x) := (x-1)^s \frac{1}{n} T_n'(x)$, we have

$$\|f^{(k)}\| = f^{(k)}(1), \quad k \ge 2s, \qquad \|g^{(k)}\| = g^{(k)}(1), \quad k \ge 2s-1,$$

therefore, since those f and g are exactly the snake-polynomials for $\mu_m(x) = (1-x^2)^{m/2}$ for $m = 2s$ and $m = 2s-1$, we can refine result of Pierre and Rahman:

$$M_{k,\mu_m} = \omega_\mu^{(k)}(1), \qquad k \ge m\,.$$

It is easy to find that $f^{(k)}(1) = \mathcal{O}(n^{2(k-s)})$ and $g^{(k)}(1) = \mathcal{O}(n^{2(k-s)+1})$, hence $\omega_\mu^{(k)}(1) = \mathcal{O}(n^{2k-m})$, in particular,

$$M_{m,\mu_m} = \omega_\mu^{(m)}(1) = \mathcal{O}(n^m)\,, \tag{9.2}$$

and that proves (9.1) for $k = m$. For $k < m$, we observe that

$$\mu_m \le \mu_k \quad \Rightarrow \quad M_{k,\mu_m} \le M_{k,\mu_k} \overset{(9.2)}{=} \mathcal{O}(n^k)\,,$$

and that completes the proof. □

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